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EFFECT OF SHEARING FORCE AND TILTING MOMENT ON A CYLINDRICAL PUNCH ATTACHED TO A TRANSVERSELY ISOTROPIC HALF-SPACE

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The effect of shearing force and tilting moment on a rigid punch, circular in plan view, and attached to a transversely isotropic half-space is considered. The distribution of shear and normal stresses under the punch is derived. Formulas are obtained which relate the angular and linear displacements of the punch to the magnitude of the shearing force and tilting moment. Let us point out that the nonaxisymmetric case of a cylindrical punch attached to an ispotropic half-space is solved in [1].

1. Let us consider a circular punch with radius a attached to a transversely isotropic half-space $z \geqslant 0$. Let the punch be subjected to a shearing force T, directed along the x-axis, and a tilting moment M. Without restricting the general nature of the problem, we may assume that the moment is directed along the y-axis. Our problem is to determine the stresses under the punch as well as the two displacements of the punch: translation u_0 and rotation δ.

Let us introduce complex displacements $u = u_x + iu_y$ and complex shear stresses $\tau = \tau_{zx} + i\tau_{yz}$ both in the plane $z = 0$. Making use of the results obtained in [2] we can write the expressions that determine the displacements of a point, having cylindrical coordinates $(\rho, \varphi, 0)$ under the action of a concentrated force, with projections P_x, P_y, P_z, applied at point $(\rho_0, \varphi_0, 0)$

$$u = \frac{1}{2R}\left\{G_1(P_x + iP_y) + G_2(P_x - iP_y)\frac{\rho e^{i\varphi} - \rho_0 e^{i\varphi_0}}{\rho e^{-i\varphi} - \rho_0 e^{-i\varphi_0}}\right\} - \frac{P_z H\alpha}{\rho e^{-i\varphi} - \rho_0 e^{-i\varphi_0}}$$

$$w = H\alpha \operatorname{Re}\left\{\frac{P_x + iP_y}{\rho e^{i\varphi} - \rho_0 e^{i\varphi_0}}\right\} + \frac{P_z H}{R} \tag{1.1}$$

Here

$$G_1 = \beta + \gamma_1\gamma_2 H, \quad G_2 = \beta - \gamma_1\gamma_2 H$$

$$H = \frac{(\gamma_1 + \gamma_2) A_{11}}{2\pi (A_{11}A_{33} - A_{13}^2)}, \quad \alpha = \frac{\sqrt{A_{11}A_{33}} - A_{13}}{A_{11}(\gamma_1 + \gamma_2)}, \quad \beta = \frac{\gamma_3}{2\pi A_{44}}$$

$$\gamma_{1,2}^2 = N \pm \sqrt{N^2 - A_{33}/A_{11}}, \quad \gamma_3 = \sqrt{A_{44}/A_{66}}$$

$$N = (A_{11}A_{33} - A_{13}^2 - 2A_{13}A_{44}) / 2A_{11}A_{44}$$

$$R^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0)$$

Here $A_{11}, A_{13}, \ldots, A_{66}$ are the elastic constants of the material of the half-space [3].

Formulas (1.1) and the law of superposition make it possible to write the integral equations of the fundamental composite problem as follows:

$$u = \frac{G_1}{2}\int_0^a\int_0^{2\pi} \frac{\tau(\rho_0, \varphi_0)}{R}\rho_0 d\rho_0 d\varphi_0 + \frac{G_2}{2}\int_0^a\int_0^{2\pi} \frac{\rho e^{i\varphi} - \rho_0 e^{i\varphi_0}}{\rho e^{-i\varphi} - \rho_0 e^{-i\varphi_0}} \frac{\bar{\tau}(\rho_0, \varphi_0)}{R}\rho_0 d\rho_0 d\varphi_0 -$$

$$- H\alpha \int_0^a\int_0^{2\pi} \frac{\sigma(\rho_0, \varphi_0)\rho_0 d\rho_0 d\varphi_0}{\rho e^{-i\varphi} - \rho_0 e^{-i\varphi_0}} \qquad (1.2)$$

$$w = H\alpha \operatorname{Re} \int_0^a\int_0^{2\pi} \frac{\tau(\rho_0, \varphi_0)\rho_0 d\rho_0 d\varphi_0}{\rho e^{i\varphi} - \rho_0 e^{i\varphi_0}} + H\int_0^a\int_0^{2\pi} \frac{\sigma(\rho_0, \varphi_0)}{R}\rho_0 d\rho_0 d\varphi_0$$

Let us express the given and the sought quantities u, w, τ, σ as series

$$Z = \sum_{n=-\infty}^{\infty} Z_n(\rho) e^{in\varphi}, \qquad Z = u, w, \tau, \sigma \qquad (1.3)$$

Since in our problem $u = u_0$ and $w = \delta\rho\cos\varphi$ for $\rho \leqslant a$, Eqs. (1.2) can be satisfied if it is assumed that only τ_0, τ_2, σ_1, σ_{-1} are different from zero.

We make use of the following easily checked identity:

$$\frac{1}{R} = \begin{cases} V(\rho, \rho_0) & (\rho_0 > \rho) \\ V(\rho_0, \rho) & (\rho_0 < \rho) \end{cases} \qquad (1.4)$$

$$V(\rho, \rho_0) = \frac{2}{\pi}\int_0^\rho \left\{2\operatorname{Re}\left[1 - \frac{x^2}{\rho\rho_0} e^{i(\varphi - \varphi_0)}\right]^{-1} - 1\right\} \frac{dx}{\sqrt{\rho^2 - x^2}\sqrt{\rho_0^2 - x^2}}$$

Let us substitute (1.3) and (1.4) into (1.2) and integrate with respect to φ_0, having changed the order of integration. Equating identical harmonics we obtain the integral equations of our problem

$$\frac{2G_1}{\rho^2}\int_0^\rho \frac{x^4 dx}{\sqrt{\rho^2 - x^2}}\int_x^a \frac{\tau_2(s)\, ds}{s\sqrt{s^2 - x^2}} + \frac{2G_2}{\rho^2}\int_0^\rho \frac{\rho^2 - 2x^2}{\sqrt{\rho^2 - x^2}} dx \int_x^a \frac{\bar{\tau}_0(s)\, s ds}{\sqrt{s^2 - x^2}} - \qquad (1.5)$$

$$- 2\pi \frac{H\alpha}{\rho^2}\int_0^\rho \sigma_1(x) x^2 dx = 0$$

$$2G_2\int_0^\rho \frac{dx}{\sqrt{\rho^2 - x^2}}\int_x^a \frac{s^2 - 2x^2}{s\sqrt{s^2 - x^2}}\bar{\tau}_2(s)\, ds +$$

$$+ 2G_1\int_0^\rho \frac{dx}{\sqrt{\rho^2 - x^2}}\int_x^a \frac{\tau_0(s)\, s ds}{\sqrt{s^2 - x^2}} - 2\pi H\alpha \int_\rho^a \sigma_{-1}(x)\, dx = u_0$$

$$2\pi H\alpha \operatorname{Re}\left\{e^{-i\varphi}\int_0^\rho \tau_0(x)\frac{x}{\rho}\,dx - e^{i\varphi}\int_\rho^a \tau_2(x)\frac{\rho}{x}\,dx\right\} + \qquad \text{(cont.)}$$

$$+\frac{4H}{\rho}\int_0^\rho \frac{x^2dx}{\sqrt{\rho^2-x^2}}\int_x^a \frac{\sigma_1(s)e^{i\varphi}+\sigma_{-1}(s)e^{-i\varphi}}{\sqrt{s^2-x^2}}\,ds = \delta\rho\cos\varphi$$

By inspection of the form of the right-hand sides of Eqs. (1.5) and allowing that the normal stress under the punch must be a real function, we can assume that

$$\tau_0 = \bar{\tau}_0,\quad \tau_2 = \bar{\tau}_2,\quad \sigma_1 = \sigma_{-1}$$

2. We shall try to find the solution of the system of equations (1.5) in the following form:

$$\sigma_1(s) = \sigma_{-1}(s) = \frac{1}{s}\int_0^s \frac{t\,df(t)}{\sqrt{s^2-t^2}} \qquad (2.1)$$

$$\tau_0(s) = \bar{\tau}_0(s) = C\int_s^a \frac{df(t)}{\sqrt{t^2-s^2}} + \frac{D}{\sqrt{a^2-s^2}}$$

$$\tau_2(s) = \bar{\tau}_2(s) = \frac{C}{s^2}\int_s^a \frac{2t^2-s^2}{\sqrt{t^2-s^2}}\,df(t) - D\,\frac{2a^2-s^2}{s^2\sqrt{a^2-s^2}}$$

Here $f(t)$ is some required function, and C and D are some constants which will be defined when the problem is solved. Substituting (2.1) into the first and second equation in (1.5), we find that they are satisfied identically if the following conditions are fulfilled:

$$C = -\frac{\alpha}{\gamma_1\gamma_2}, \qquad D = \frac{C}{a}\int_0^a t\,df(t) \qquad (2.2)$$

$$\pi^2 D\beta + 2\pi H\alpha\int_0^a \frac{f(t)\,dt}{\sqrt{a^2-t^2}} = u_0 \qquad (2.3)$$

Substituting (2.1) into the third equation of (1.5), makes it look as follows:

$$\frac{8H}{\rho}\int_0^\rho \frac{x\,dx}{\sqrt{\rho^2-x^2}}\int_0^a \ln\frac{a\sqrt{|x^2-t^2|}}{|x\sqrt{a^2-t^2}-t\sqrt{a^2-x^2}|}\,df(t) +$$

$$+2\pi\frac{H}{\rho}\frac{\alpha^2}{\gamma_1\gamma_2}\left[2\int_\rho^a \sqrt{t^2-\rho^2}\,df(t) - \int_0^a t\,df(t)\right] + 2\pi H\alpha\frac{a}{\rho}D = \delta\rho$$

Integrating this expression by parts, applying (2.2), multiplying the obtained result by $\rho^2(r^2-\rho^2)^{-1/2}$, and integrating with respect to ρ between 0 and r gives us the equation for the required $f(t)$

$$\sqrt{a^2-r^2}\int_0^a \Phi(r,t)\,dt - \lambda^2\int_0^a \Psi(r,t)\,dt = \frac{\delta}{2\pi H} \qquad (2.4)$$

$$\Phi(z,t) = \frac{f(t)}{\sqrt{a^2-t^2}\,(t^2-z^2)}, \qquad \lambda = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}$$

$$\Psi(z,t) = \sqrt{a^2-t^2}\,\Phi(z,t) \qquad (2.5)$$

We introduce the following notation:

$$Y_{c,s}(z) = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left[\theta\ln\frac{a+z}{a-z}\right]$$

The quantity θ will be defined later on.

Let us find the general solution of Eq. (2.4) for the case, when the right-hand side is arbitrary and equals $\chi(r)$. We multiply both sides by $(r^2 - x^2)^{-1}\, Y_c(r)$ and integrate over r between 0 and a, making use of the Poincaré-Bertrand formula [4], we have

$$-\frac{\pi^2}{4x^2}(1-\lambda^2)\, f(x)\, Y_c(x) + \frac{\pi}{2}\operatorname{th}\pi\theta \int_0^a \Phi(x,t)\left[\frac{\sqrt{a^2-t^2}}{t}Y_s(t) - \frac{\sqrt{a^2-x^2}}{x}Y_s(x)\right]dt - \frac{\pi}{2}\lambda^2 \operatorname{cth}\pi\theta \int_0^a \Psi(x,t)\left[\frac{Y_s(t)}{t} - \frac{Y_s(x)}{x}\right]dt = X_c(x)$$

Here and in what follows

$$X_{c,s}(x) = \int_0^a \frac{\chi(r)}{r^2 - x^2}\begin{Bmatrix} Y_c(r) \\ rY_s(r)\end{Bmatrix} dr$$

We find the quantity θ from the condition

$$\operatorname{th}\pi\theta - \lambda^2 \operatorname{cth}\pi\theta = 0, \quad \theta = \pi^{-1}\operatorname{Arth}\lambda \tag{2.6}$$

The obtained rather cumbersome expression simplifies and becomes

$$-\frac{\pi^2 f(x)}{4x^2 \operatorname{ch}^2\pi\theta} Y_c(x) + \frac{\pi}{2}\operatorname{th}\pi\theta\,\frac{Y_s(x)}{x}\left[-\sqrt{a^2-x^2}\int_0^a \Phi(x,t)\,dt + \int_0^a \Psi(x,t)\,dt\right] = X_c(x) \tag{2.7}$$

When both sides of Eq. (2.4) are multiplied by $r(r^2 - x^2)^{-1} Y_s(r)$ and integrated over r from 0 to a, we obtain

$$-\frac{\pi^2 f(x)}{4x \operatorname{ch}^2\pi\theta} Y_s(x) + \frac{\pi}{2}\operatorname{th}\pi\theta\, Y_c(x)\left[\sqrt{a^2-x^2}\int_0^a \Phi(x;t)\,dt - \int_0^a \Psi(x,t)\,dt\right] = X_s(x) \tag{2.8}$$

From (2.7) and (2.8) we find now

$$f(x) = -\frac{4\operatorname{ch}^2\pi\theta}{\pi^2}\, x\left[xY_c(x)\, X_c(x) + Y_s(x)\, X_s(x)\right] \tag{2.9}$$

Complete solution of Eq. (2.4) can be obtained by adding to (2.9) a term in the form of $AY_c(x)$ which represents the solution of the homogeneous equation, where A is an arbitrary constant.

3. We apply now the hitherto obtained results to our problem. We have then

$$f(t) = \frac{\delta\operatorname{ch}^2\pi\theta}{\pi^2 H}\left[xY_s(x) + a\theta Y_c(x)\right] + AY_c(x) \tag{3.1}$$

Substituting (3.1) into (2.2) and (2.3) and integrating, we can find the parameters

$$A = \left(u_0 - \frac{\delta a\theta\alpha}{\operatorname{th}\pi\theta}\right)\left[\frac{\pi^2 H\alpha}{\operatorname{ch}\pi\theta}\left(1 + \frac{\beta}{\gamma_1\gamma_2 H}\,\frac{\pi\theta}{\operatorname{th}\pi\theta}\right)\right]^{-1}, \qquad D = \frac{\alpha}{\gamma_1\gamma_2}\,\frac{\pi\theta}{\operatorname{sh}\pi\theta}A \tag{3.2}$$

At this stage, formulas (1.4), (2.1), (3.1) and (3.2) completely determine the stresses under the punch. All that remains to do is to find the relations between the displacements of the punch and the external forces applied to it.

We make use of the conditions of statics

$$T = 2\pi \int_0^a \tau_0(\rho)\,\rho d\rho, \qquad M = \int_0^a \int_0^{2\pi} (\sigma_1 e^{i\varphi} + \sigma_{-1} e^{-i\varphi})\,\rho^2 \cos\varphi\, d\rho\, d\varphi$$

After carrying out all the calculations, we obtain

$$T = 4\pi^2 \frac{a\theta}{\operatorname{sh}\pi\theta}\frac{\alpha A}{\gamma_1\gamma_2}, \qquad M = \frac{4\delta a^3\theta}{3H \operatorname{th}\pi\theta}(1+\theta^2) - \frac{4\pi^2 a^2\theta^2}{\operatorname{ch}\pi\theta} A \tag{3.3}$$

Expressions (3.2) and (3.3) enable us to determine the displacements of the punch

$$u_0 = \left[\frac{\pi\beta}{4a} + \frac{\gamma_1\gamma_2 H \operatorname{th}\pi\theta}{4a\theta(\theta^2+1)}(4\theta^2+1)\right] T + \frac{3H\alpha}{4a^2(\theta^2+1)} M$$

$$\delta = \frac{3H\alpha}{4a^3\theta(\theta^2+1)}\left[a\theta T + \frac{M}{\sqrt{\gamma_1\gamma_2}}\right] \tag{3.4}$$

In the case of an isotropic half-space the formulas (3.4) yield a solution which is in agreement with that given in [1].

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EQUATIONS OF THE THEORY OF PERFECT PLASTICITY IN TERMS OF THE COMPONENTS OF DISPLACEMENT VELOCITIES

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The equations of the theory of perfect plasticity are derived in terms of the components of displacement velocities. These equations are analogous to the Lamé equations in the theory of elasticity, when displacements are treated as the unknowns.

In the theory of perfect plasticity the components of stress can be expressed in terms of the components of displacement velocities by means of the following formulas [1, 2]:

$$\sigma_{ij} = \partial D / \partial e_{ij} \tag{1}$$

where $D = D(e_{ij})$ is the dissipation function, and e_{ij} are the components of the velocity of plastic deformation (for the sake of simplicity, the body is assumed rigidly plastic).

Substituting expressions (1) into the equations of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{2}$$